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Diffraction of surface waves on an incompressible fluid

By W. E. WILLIAMS
Department of Applied Mathematics, Liverpool University

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The problem considered is that of the diffraction of the field of a point source in a fluid of infinite depth by an infinite vertical cylinder. It is shown that the surface wave component of the velocity potential may be expressed in terms of the solution to a classical electromagnetic (or acoustic) diffraction problem.

1. Introduction

The present paper examines the problem of the diffraction of the field of a periodically pulsating point source in a fluid of infinite depth by a vertical cylinder S extending from a finite distance above the fluid surface to an infinite distance below it. (Small amplitude oscillations only are considered.) For the particular case when the cylinder is an infinitely thin half plane solutions have been obtained by Voit (1961) and Levine (1963). The analysis of both papers is rather complicated but the final form of Levine's solution is comparatively simple. In particular he shows that the surface wave component of the velocity potential can be expressed in terms of the solution of the problem of the diffraction by the half plane of the field of a magnetic (or acoustic) line source parallel to the edge. The general relationship between problems of electromagnetic diffraction and those of diffraction of water waves is of course well known (cf. Wehausen & Laitone 1960), but the particular relationship derived by Levine seems to be new.

In the present work conventional elementary transform methods are employed to obtain the solution in a form similar to that given by Levine for the half-plane problem. A generalization of Levine's result concerning the surface wave component is obtained, namely that this component arises entirely from a function which is the solution of a particular electromagnetic diffraction problem. The relevant problem is that of the diffraction of the field of a magnetic (or acoustic) line source by an infinite cylinder whose generators are parallel to the line source and whose cross-section in planes normal to the generators is identical with that of S.

2. Detailed formulation and solution of the problem

A cartesian co-ordinate system $Oxyz$ is chosen with its origin in the mean free surface and such that the fluid occupies the region $z \geqslant 0$. The diffracting object is assumed to be a rigid cylinder S of uniform cross-section whose generators are parallel to the z-axis and which extends from some finite distance above the mean free surface to an infinite distance below it. The curve bounding a section of S

in planes normal to the z-axis will be denoted by C. It is assumed that the exciting field is due to a point source at (x_0, y_0, z_0). The velocity potential describing the motion will be defined by $\phi e^{-i\omega t}$, where

$$\nabla^2\phi = -\delta(x-x_0)\,\delta(y-y_0)\,\delta(z-z_0). \tag{1}$$

The normal derivative of $\phi(\partial\phi/\partial n)$ on S has to vanish and on $z = 0$ we have that

$$\partial\phi/\partial z = -k\phi, \quad k = \omega^2/g.$$

ϕ has to tend to zero as z becomes infinite and also to represent a disturbance travelling away from S.

The boundary condition on $z = 0$ is similar to one occurring in heat conduction problems and it thus seems appropriate to attempt a solution by employing a transform method used in the solution of these latter problems (Churchill 1958). The appropriate transform Φ is defined by

$$\Phi(x,y,\alpha) = \int_0^\infty (-k\sin\alpha z + \alpha\cos\alpha z)\,\phi(x,y,z)\,dz. \tag{2}$$

For k negative equation (2) has a unique inverse given by $\phi = \mathscr{L}\Phi$, where the operator $\mathscr{L}$ is defined by

$$\mathscr{L}\Phi = \frac{2}{\pi}\int_0^\infty \frac{(-k\sin\alpha z + \alpha\cos\alpha z)}{(\alpha^2+k^2)}\,\Phi(x,y,\alpha)\,d\alpha.$$

For positive k, however, it follows by direct manipulation that

$$\mathscr{L}\Phi = \phi - 2k\int_0^\infty e^{-k(z+t)}\phi(x,y,t)\,dt, \tag{3}$$

and it thus follows immediately from equation (3) that for k positive the inverse of equation (2) is $\mathscr{L}\Phi + f(x,y)\,e^{-kz}$ where f is an arbitrary function.

It follows from equation (1) by conventional transform methods that Φ is equal to $(k\sin\alpha z_0 - \alpha\cos\alpha z_0)\,\psi(x,y,\alpha)$, where

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} - \alpha^2\psi = \delta(x-x_0)\,\delta(y-y_0). \tag{4}$$

The conditions to be satisfied by ϕ are clearly satisfied if we determine a solution ψ of equation (4) with $\partial\psi/\partial n = 0$ on C and $\psi \to 0$ as $(x^2+y^2)^{\frac{1}{2}} \to \infty$. This boundary-value problem for ψ is of a classical type and a solution is known in closed form for some particular curves C (e.g. circle, semi-infinite line, sector).

Inversion now gives

$$\phi = -\frac{1}{\pi}\int_0^\infty [\cos\alpha(z-z_0) + \cos\alpha(z+z_0)]\,\psi(x,y,\alpha)\,d\alpha$$
$$+\frac{2k}{\pi}\int_0^\infty \frac{[k\cos\alpha(z+z_0) + \alpha\sin(z+z_0)]\,\psi(x,y,\alpha)}{k^2+\alpha^2}\,d\alpha + f(x,y)\,e^{-kz}. \tag{5}$$

The first integral in equation (5) is clearly the solution ϕ of equation (1) for the particular case $k = 0$ and hence, if $G(x,y,z)$ denotes the solution of equation (1) with $\partial\phi/\partial n$ vanishing on an infinite cylinder which intersects planes normal to the z-axis in a curve C, it follows that

$$\phi = G(x,y,z)+G(x,y,-z)+\frac{2k}{\pi}\int_0^\infty \frac{[k\cos\alpha(z+z_0)+\alpha\sin(z+z_0)]\,\psi\,d\alpha}{k^2+a^2}+f(x,y)\,e^{-kz}. \tag{6}$$

It now follows from equations (1), (4) and (6) that f is equal to $-2ke^{-kz_0}\chi(x,y)$, where

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+k^2\right)\chi = \delta(x-x_0)\,\delta(y-y_0). \tag{7}$$

The function χ is to represent an outgoing wave as $(x^2+y^2)^{\frac{1}{2}}\to\infty$ and $\partial\chi/\partial n$ is to vanish on C. This boundary-value problem for χ is a classical one in diffraction theory and a solution is known for particular cases of C. In the notation of electromagnetic theory χ represents the total magnetic field component in the z direction in the region external to an infinite conducting cylinder of cross-section C when the incident magnetic field is due to a magnetic line source at (x_0, y_0) such that its magnetic field in free space is $-\frac{1}{4}iH_0^{(2)}[k\{r^2+r_0^2-2rr_0\cos(\theta-\theta_0)\}^{\frac{1}{2}}]$, where

$$x = r\cos\theta,\quad y = r\sin\theta,\quad x_0 = r_0\cos\theta_0,\quad y_0 = r_0\sin\theta_0.$$

Though the general form of the velocity potential in equation (6) is rather complicated, it is possible to deduce significant results from it concerning the surface wave component of the fluid motion. The terms involving G are purely static, the integral in equation (6) tends to zero exponentially as $r\to\infty$ and thus the surface wave component for large r arises entirely from the term involving χ. It follows from electromagnetic diffraction theory that as $r\to\infty$

$$\chi \sim (2/\pi kr)^{\frac{1}{2}}\exp\{i(kr-\tfrac{1}{4}\pi)\}\,A(r_0,\theta,\theta_0),$$

where A is known as the far field amplitude of χ. If we now assume that $r_0\to\infty$, it follows from the reciprocity properties of the Green's function that

$$\chi \sim (2/\pi kr_0)^{\frac{1}{2}}\exp\{i(kr_0-\tfrac{1}{4}\pi)\}\,A(r,\theta,\theta_0).$$

Thus on letting r_0 become infinite in the expression for the incident field it follows that $A(r,\theta,\theta_0)$ is the total magnetic field at the point (r,θ) due to the incident plane wave $\frac{1}{4}i\exp\{ikr\cos(\theta-\theta_0)\}$. Thus the amplitude of the surface wave term is directly related to the solution of a plane wave diffraction problem and for certain particular curves C a considerable amount of information is available concerning the solution of such problems (cf. Jones 1964, ch. 8, 9). A further simplification occurs if r_0 is large; we then have that

$$\chi \sim \tfrac{1}{2}(1/2\pi kr_0)^{\frac{1}{2}}\exp\{-i(kr_0+3\pi/4)\}\times\begin{bmatrix}\text{solution of diffraction problem due}\\ \text{to plane wave }\exp\{ikr\cos(\theta-\theta_0)\}\end{bmatrix}.$$

Thus in this case the far field behaviour of χ is directly related to the far field behaviour of the solution of a plane wave diffraction problem. For the particular case when C is (a) a circle of radius a, (b) the parabola $-y^2 = 2\xi_0^2x-\xi_0^4$, and (c) a line of width b, explicit forms are known for the far field pattern of the plane wave diffraction problem for large and small values of the parameters ka, kb, $k\xi_0$ (cf. Jones 1964; Seshadri 1959).

We shall examine in slightly more detail the particular case when C is the infinite sector of angle $m\pi$. The exact solution χ for this problem is known (Jones 1964) and χ will consist of two parts one of which arises from the incident and reflected fields (i.e. the terms of geometrical optics) and the other is a diffracted term (i.e. an edge effect term). It may be deduced immediately from the known form of χ that as r, $r_0 \to \infty$ the contribution of the diffracted term to ϕ is given near the surface by

$$\phi_{\text{surface}} \sim \frac{i \sin(\pi/m)\, e^{-kz_0}}{2m(rr_0)^{\frac{1}{2}}} \exp\{i[k(r+r_0)-\omega t]\}$$
$$\times \left[\frac{1}{\cos(\pi/m)-\cos\{(\theta-\theta_0)/m\}}+\frac{1}{\cos(\pi/m)-\cos\{(2\pi+\theta+\theta_0)/m\}}\right]. \qquad (8)$$

For $m = 2$ equation (8) can, when notational changes are taken into account, be identified with a corresponding result given by Voit.

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